Lecture 9: Solution Methods Using Separation of Variables

- Consider the case $\frac{\partial u}{\partial t} = a(t)b(x)\Delta u$. We can rewrite this as
  $\frac{1}{a(t)}\frac{\partial u}{\partial t} = b(x)\Delta u$ when $a(t) \neq 0$.
  "splitting" the equation into parts depending on t & x.

- In such cases, we use an ansatz $u(t,x) = v(t)\Phi(x)$, reducing the PDE to some pair of PDEs. Using further symmetries, we can separate spatial variables in higher dimensions, giving only ODEs. In either case, we simplify our equation of interest.

Helmholtz Equation

- The classical evolution equations on $\mathbb{R}^n$ have the form

(A) $\quad P_t u - \Delta u = 0$

for P_t a first or second-order differential operator only in time.
e.g.) Heat, wave, Schrodinger

$\boxed{\text{Lemma 5.1}}$ If u is a classical solution of 5.1 of the form $u(t,x) = v(t)\Phi(x)$ for $t \in \mathbb{R}$ and ~~...~~ $x \in U \subseteq \mathbb{R}^n$, then in any region where u is nonzero there is a constant κ such that

$$P_t v = \kappa v \qquad \Delta\Phi = \kappa\Phi$$

$\boxed{\text{Pf}}$ Substituting $u = v\Phi$ into (A) gives $\Phi P_t v - v\Delta\Phi = 0$. Assuming u is nonzero, $\frac{1}{v}P_t v = \frac{1}{\Phi}\Delta\Phi$ by dividing by u. Since the LHS is only in t & the RHS is only in x, both sides must be some constant κ. □

-) The two equations are analogous to eigenvalue equations from linear algebra. (P_t & Δ are operators in this POV)

-) The spatial problem

(B) $\quad -\Delta\Phi = \lambda\Phi$

is called the Helmholtz equation, where a negative is added so $\lambda \geq 0$ for most common types of boundary conditions.

This is the Laplace eigenvalue equation
Φ = eigenfunction
λ = eigenvalue

We focus only on one spatial variable for the moment.

$\boxed{\text{Th}^{\underline{m}}\ 5.2}$ For $\Phi \in C^2[0,\ell]$, the equation $-\frac{d^2\Phi}{dx^2} = \lambda\Phi$, $\Phi(0) = \Phi(\ell) = 0$, has nonzero solutions iff. $\lambda_n = \frac{\pi^2 n^2}{\ell^2}$, for $n \in \mathbb{N}$. Up to constant multiplication, the solutions are $\Phi_n(x) = \sin(x\sqrt{\lambda_n})$.

$\boxed{\text{Pf}}$ The PDE implies $\lambda \int_0^\ell |\Phi|^2 dx = -\int_0^\ell \frac{d^2\Phi}{dx^2}\bar{\Phi}\, dx$ $\quad (|\Phi|^2 = \Phi\bar{\Phi})$.

Next, $-\int_0^\ell \frac{d^2\Phi}{dx^2}\bar{\Phi}\,dx = \int_0^\ell \frac{d\Phi}{dx}\frac{d\bar{\Phi}}{dx}\,dx = \int_0^\ell \left|\frac{d\Phi}{dx}\right|^2 dx.$

If Φ isn't identically 0, $\lambda \geq 0$. If $\lambda = 0$, $\left|\frac{d\Phi}{dx}\right| = 0$ everywhere so that Φ is constant, and $\Phi \equiv 0$.

If $\lambda > 0$, $\Phi'' + \lambda\Phi = 0$ gives a general solution $\Phi(x) = C_1\sin(x\sqrt{\lambda}) + C_2\cos(x\sqrt{\lambda})$. The boundary conditions give $\Phi(x) = C_1\sin(x\sqrt{\lambda})$ ① and $\sin(\ell\sqrt{\lambda}) = 0$ ② so that $\ell\sqrt{\lambda} \in \pi\cdot\mathbb{N}$ or $\sqrt{\lambda} = \frac{\pi n}{\ell}$. $\square$

- For the string model, recall that $c = \sqrt{T/\rho}$ and the equation is
$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0 \qquad u(t,0) = u(0,\ell) = 0$$

→ the spatial solution is given above, and the temporal ~~solution~~ eigenvalue equation is $-\frac{\partial^2 V}{\partial t^2} = c^2\lambda_n V$ with solution
$$V_n(t) = a_n e^{i\omega_n t} + b_n e^{-i\omega_n t} \qquad \text{for } a_n, b_n \in \mathbb{C}$$
and $\omega_n = c\sqrt{\lambda_n} = \frac{cn\pi}{\ell}$

→ To make solutions real-valued, we need $\bar{a}_n = b_n$

Combining the solutions
$$u_n(t,x) = \left[a_n e^{i\omega_n t} + b_n e^{-i\omega_n t}\right]\sin(x\sqrt{\lambda_n}) \quad \text{for } n \in \mathbb{N}$$

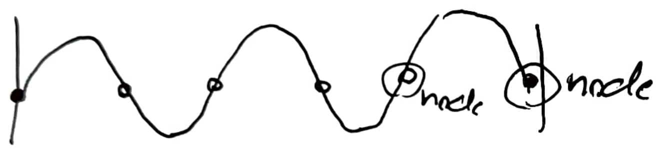
- These are called "pure tone" solutions b/c they model oscillation at a single frequency ω_n. For light waves, the frequency is a color! Thus, $\{\omega_n\}$ is often called a "spectrum" (a general term for eigenvalues).

- From $\omega_n = \frac{c\pi n}{\ell}$, we can deduce the fundamental frequency of a string

To convert to Hertz, $\frac{\omega_n}{2\pi} = \frac{cn}{2\ell} = \frac{n}{2\ell}\sqrt{T/\rho}$ ⇝ "Mersenne's Law"

$\frac{\omega_1}{2\pi} = \frac{1}{2\ell}\sqrt{T/\rho}$ is called the fundamental frequences, higher n give "overtones"

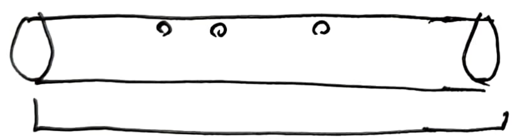
- The points at which the string stays stationary are called nodes



n^{th} frequency has $n+1$ nodes

touching a string at a node knocks out lower frequencies - called a "harmonic"

ex.) Let us use the 1D wave equation to model air pressure fluctuations in a clarinet



⇝ length parameter x

$u(t,x)$ = pressure at x at time t (above atmospheric, so $u=0$ is atmospheric)

Max pressure occurs at the mouthpiece $x=0$. A local max is a critical point of $u(t,\cdot)$, so $\frac{\partial u}{\partial x}(t,0) = 0$ is a B.C.

At the other end, pressure doesn't fluctuate (open to air) and $u(t,\ell)=0$

Thus, the wave equation separates into Helmholtz Problem

$$-\frac{d^2\phi}{dx^2} = \lambda\phi, \quad \phi'(0)=0, \quad \phi(\ell)=0$$

giving solution

$$\phi(x) = c_1 \sin(x\sqrt{\lambda}) + c_2\cos(x\sqrt{\lambda})$$

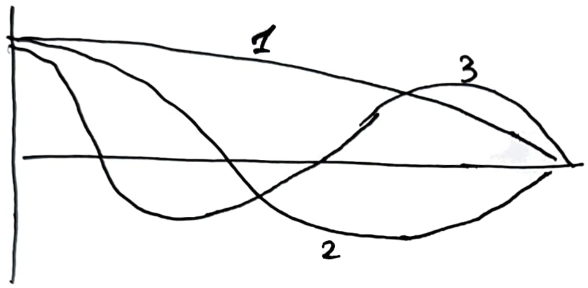
The first B.C. $\phi'(0)=0$ gives

$$\phi(x) = c_2 \cos(x\sqrt{\lambda})$$

and $\phi(\ell)=0$ gives $\lambda_n = \frac{\pi^2}{\ell^2}(n-\frac{1}{2})^2$ for some n

($\ell\sqrt{\lambda} = n\cdot\frac{\pi}{2}$ so $\lambda = \left(\frac{n\pi}{2\ell}\right)^2$, and we shift to account for n=0)

eigenfunctions for $c_2=1$



The corresponding oscillation frequencies are

$$\omega_n = \frac{c\pi}{\ell}(n-\frac{1}{2})$$

$\Rightarrow$ This predicts only odd multiples of ω_1 - imperfect model.